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Symmetries and Compactifications of (4,0) Conformal Gravity

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ABSTRACT

The free (4,0) superconformal theory in 6 dimensions and its toroidal dimensional reductions are studied. The reduction to four dimensions on a 2-torus has an $SL(2,\mathbb{Z})$ duality symmetry that acts non-trivially on the linearised gravity sector, interchanging the linearised Einstein equations and Bianchi identities and giving a self-duality between strong and weak coupling regimes. The possibility of this extending to an interacting form of the theory is discussed and implications for the non-geometric picture of gravity that could emerge are considered.

1. Introduction

In [1], it was proposed that M-theory could have a superconformal phase in $D = 6$ dimensions with (4,0) supersymmetry and $OSp^*(8/8)$ superconformal invariance, arising from a strong coupling limit of five dimensional string theory (i.e. M-theory compactified on a 6-torus). The five-dimensional theory with $D = 5$ Planck length l is the six-dimensional conformal theory compactified on a circle of radius $r = l$, so that a limit in which l becomes large is a decompactification limit in which a new 6th dimension opens up. The $D = 5$ Planck scale would thus have a geometric origin.

The free (4,0) theory in six dimensions arises as a limit of the free $D = 5$ linearised supergravity theory [1]. This, together with the analogous strong coupling limit of $D = 5, N = 4$ super-Yang-Mills to an interacting (2,0) superconformal field theory, motivated the conjecture of [1] that there could be a strong coupling limit of the interacting $D = 5$ supergravity theory to something that reduces to the free $D = 6$ (4,0) theory in the linearised limit. A major obstacle in the checking of these conjectures is that the interacting (2,0) or (4,0) superconformal field theories – if they exist – cannot be formulated as local covariant field theories. One of the aims here is to seek indirect evidence for these conjectures by investigating some of the consequences they would have, if true.

The (4,0) supermultiplet in $D = 6$ has a 4th rank tensor gauge field $C_{MN\,PQ}$ which has the same algebraic symmetry properties as the Riemann tensor, and reducing to $D = 5$ gives a graviton $h_{\mu\nu}$ only, with no other fields, and the radius r sets the $D = 5$ Planck scale. The (4,0) multiplet reduces to the $N = 8$ supergravity multiplet in $D = 5$. The $D = 6$ theory has no graviton but has $C_{MN\,PQ}$ instead, and so the gravitational field does not have an interpretation in terms of conventional Riemannian geometry. A further reduction to $D = 4$ on a circle of radius r' to an $N = 8, D = 4$ theory is straightforward; in particular, the $D = 5$ graviton $h_{\mu\nu}$ gives a graviton, a vector and a scalar in $D = 4$, with the scalar field ϕ given in terms of the radius r'. The two radii r, r' appear to play rather different roles

here, but the final result might be expected to be independent of which dimension is compactified first and so should have a symmetry which interchanges the two radii. The ratio $g^2 = r'/r$ defines a dimensionless coupling constant (the expectation value of e^{ϕ}) and interchanging the two radii takes $g \to 1/g$, so that such a symmetry would be a duality interchanging weak and strong coupling regimes, a kind of gravitational S-duality.

More generally, the reduction on a two-torus might be expected to give a $D = 4$ theory with an $SL(2, \mathbb{Z})$ duality symmetry in addition to the $E_6(\mathbb{Z})$ which is expected to be the duality symmetry of the (4,0) theory. However, the $D = 4, N = 8$ theory has an $E_7(\mathbb{Z})$ U-duality symmetry [2] and E_7 does not have an $E_6 \times SL(2)$ subgroup, so the $SL(2, \mathbb{Z})$ duality expected from the $D = 6$ description could not be any of the known U-duality symmetries of the theory [3]. Moreover, such an $SL(2, \mathbb{Z})$ duality would necessarily act non-trivially on the graviton, while the U-duality leaves the Einstein-frame metric invariant. Such an $SL(2, \mathbb{Z})$ symmetry would then necessarily include new symmetries not contained in the U-duality group.

One possibility is that there is indeed such a new $SL(2, \mathbb{Z})$ symmetry in the full theory, requiring some modification of the supergravity description. This would then be similar to the case of the interacting (2,0) theory whose reduction to 4 dimensions has an $SL(2, \mathbb{Z})$ S-duality symmetry. This is not a symmetry of the non-abelian $N = 4$ super-Yang-Mills field equations, so that this field theory does not give a complete description of the S-dual theory. Alternatively, the (4,0) theory is not based on conventional geometry and so the geometric arguments leading to an $SL(2, \mathbb{Z})$ symmetry may be invalid. In particular, the $D = 5$ diffeomorphisms do not arise from $D = 6$ diffeomorphisms, but from a higher-spin symmetry in $D = 6$, and so the group $SL(2, \mathbb{Z})$ of large diffeomorphisms on a 2-torus need not play any role in the full theory.

In this paper, the (4,0) theory in 6 dimensions and its dimensional reduction and symmetries will be investigated. Much of the paper will be devoted to the free

(4,0) theory, which is known explicitly, and its properties have many similarities with the free (2,0) theory. The free (4,0) theory can be formulated in terms of free fields in a fixed flat spacetime background. The bosonic fields consist of the gauge field C_{MNPQ}, 27 2-form gauge fields B_{MN} with self-dual field strengths and 42 scalars. If the background spacetime is the product of 4-dimensional Minkowski space and a 2-torus, the free theory has an $SL(2,\mathbb{Z})$ symmetry arising from diffeomorphisms of the 2-torus background. For a 2-form gauge field reduced on a 2-torus, this is the S-duality of the resulting $D = 4$ Maxwell theory, and this is reviewed in section 2. In sections 3 and 4, the dimensional reduction of the gauge field C_{MNPQ} and the resulting gravitational $SL(2,\mathbb{Z})$ symmetry are studied. As well as transforming photons into dual photons as in standard electromagnetic duality, it transforms gravitons into dual gravitons. In section 5, it is shown how this occurs as part of the duality symmetry of the linearised $D = 4$ supergravity. In section 6, the question of whether these symmetries can extend to an interacting form of the theory (if it exists) is addressed, leading to some insights into the possible structure of the theory.

2. Dimensional reduction of 2-Form Gauge Fields

The abelian (2,0) theory has a self-dual 2-form gauge field and five scalars, and its dimensional reduction on a circle gives $D = 5$ super-Maxwell theory (i.e. abelian super-Yang-Mills) and the reduction on a 2-torus gives the $D = 4, N = 4$ super-Maxwell theory with an $SL(2,\mathbb{Z})$ S-duality symmetry arising from the mapping class group of the 2-torus [4]. In this section, the dimensional reduction of both unconstrained and self-dual $D = 6$ 2-form gauge fields will be reviewed, with particular attention to the emergence of the duality symmetry.

A 2-form gauge field B_{MN} in six dimensions (where $M, N = 0, 1, ..., 5$) enjoys the gauge invariance

$$\delta B_{MN} = \partial_{[M}\lambda_{N]} \tag{2.1}$$

and the invariant field strength 3-form $H = dB$ satisfies the Bianchi identity

$$dH = 0 \tag{2.2}$$

The conformally invariant six-dimensional free action

$$S = -\frac{1}{12} \int d^6x \, \sqrt{-g} H_{MNP} H^{MNP} \tag{2.3}$$

implies the field equation

$$d * H = 0 \tag{2.4}$$

(where $*$ denotes the Hodge dual, $(*H)_{MNP} = \frac{1}{6}\epsilon_{MNPQRS} H^{QRS}$ and ϵ_{MNPQRS} is a tensor, not a tensor density) and the 2-form has 6 degrees of freedom in the $(3,1) + (1,3)$ representation of the little group $SU(2) \times SU(2)$.

Imposing the self-duality constraint

$$H = *H \tag{2.5}$$

halves the number of degrees of freedom, leaving 3 physical modes in the (3,1) of $SU(2) \times SU(2)$. The self-duality condition (2.5) together with the Bianchi identity $dH = 0$ implies the field equation $d * H = 0$. The action (2.3) vanishes for self-dual H.

The dimensional reduction of a general 2-form gauge field B_{MN} from 6 to 5 dimensions on a circle of radius r gives a vector field $A_\mu = B_{\mu 5}$ with field strength $F = dA$ and a 2-form $B_{\mu\nu}$ with field strength $H = dB$ (where $\mu, \nu = 0, 1, \ldots 4$). The reduction of the action (2.3) gives

$$S = -\int d^5x \, \left(\frac{1}{4g_{YM}^2} F_{\mu\nu} F^{\mu\nu} + \frac{g_{YM}^2}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) \tag{2.6}$$

where the dimensionful $D = 5$ coupling constant g_{YM} is given by

$$g_{YM}^2 = r \tag{2.7}$$

The 2-form $B_{\mu\nu}$ can be dualised to a second vector field $\tilde{A}_\mu$ with field strength

$\tilde{F} = d\tilde{A}$ given by

$$\tilde{F} = r * H \tag{2.8}$$

so that the action becomes

$$S = -\frac{1}{4g_{YM}^2} \int d^5x \left(F_{\mu\nu}F^{\mu\nu} - \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \right) \tag{2.9}$$

Note that the Bianchi identity for H (2.2) implies the field equation for $\tilde{A}$ and vice versa. The 6 degrees of freedom of a general B_{MN} in $D = 6$ gives the 3 degrees of freedom of A plus the 3 degrees of freedom of B or $\tilde{A}$ in $D = 5$.

However, if the $D = 6$ gauge field is self-dual, then (2.5) implies the $D = 5$ constraint

$$F = r * H \tag{2.10}$$

so that B is determined by A, or equivalently $\tilde{F} = F$ so that a gauge can be chosen in which $\tilde{A} = A$. Only 3 degrees of freedom then remain in $D = 5$, which can be taken to be represented by the field A with action

$$S = -\frac{1}{4g_{YM}^2} \int d^5x \, F_{\mu\nu}F^{\mu\nu} \tag{2.11}$$

The reduction of B_{MN} from 6 to 4 dimensions on a 2-torus with constant metric γ_{ij} gives two vector fields $A_{mi} = B_{mi}$, a scalar field $\phi = \frac{1}{2}\epsilon^{ij}B_{ij}$ and a 2-form gauge field B_{mn}, where $m, n = 0, 3, 4, 5$ and $i, j = 1, 2$. (Here ϵ_{ij} has components $\epsilon_{12} = \sqrt{det\gamma_{ij}}$ and the indices i, j are raised and lowered with the metric γ_{ij}.) The 2-form B can be dualised to a second scalar $\tilde{\phi}$, defined by

$$d\tilde{\phi} = *H \tag{2.12}$$

with $H = dB$. Then reducing the $D = 6$ action (2.3) gives the $D = 4$ action

$$S = -\int d^4x \left(\frac{1}{4}\tilde{\gamma}^{ij} F_{mn\,i}F^{mn}{}_j + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\tilde{\phi})^2 \right) \tag{2.13}$$

where

$$\tilde{\gamma}^{ij} = \sqrt{\gamma}\gamma^{ij} \tag{2.14}$$

This action is manifestly $SL(2,\mathbb{R})$ invariant, with F_i transforming as

$$F \to SF \tag{2.15}$$

where $S_i{}^j$ is an $SL(2,\mathbb{R})$ matrix and γ_{ij} transforms as

$$\gamma \to S\gamma S^t \tag{2.16}$$

The boundary conditions on the 2-torus coordinates or the Dirac quantization conditions in $D=4$ break the $SL(2,\mathbb{R})$ symmetry down to $SL(2,\mathbb{Z})$.

Suppose now that B_{MN} satisfies the self-duality constraint (2.5). Dimensionally reducing (2.5) gives

$$d\phi = *H \tag{2.17}$$

and

$$\tilde{\gamma}^{ij} F_{mn\,j} = \frac{1}{2}\epsilon_{mnpq}\epsilon^{ij} F^{pq}{}_j \tag{2.18}$$

so that A_1, A_2 are dual potentials corresponding to the same electromagnetic field, and ϕ is dual to B, so that one can set $\tilde{\phi} = \phi$. The constraint (2.18) can be written in the form used in [7] as

$$F_i = J_i{}^j * F_j \tag{2.19}$$

where

$$J_i{}^j = \tilde{\gamma}_{ik}\epsilon^{kj} = \frac{1}{\sqrt{\gamma}}\gamma_{ik}\epsilon^{kj} \tag{2.20}$$

and satisfies $J^2 = -1$. The $3+3$ degrees of freedom contained in A_1, ϕ, A_2, B then satisfy 3 constraints, leaving 3 independent degrees of freedom which can be taken to be represented by A_1, ϕ.

3. Tensor Gauge Fields

3.1. The Graviton

It will be useful to review some of the properties of a free symmetric tensor gauge field h_{MN} in D dimensions before generalising to the fourth rank gauge fields of the type that occur in the (4,0) multiplet. It is useful to write the spacetime metric as

$$g_{MN} = \eta_{MN} + \lambda h_{MN} \tag{3.1}$$

and the Einstein action as

$$S = \frac{1}{\lambda^2 l^{D-2}} \int d^D x \sqrt{-g} R \tag{3.2}$$

where λ is a dimensionless coupling constant. If $\lambda \neq 0$, then it can be absorbed into the definitions of h_{MN} and l. The two kinds of limit discussed in this paper are the free limit in which $\lambda \to 0$ leaving a free theory with a quadratic action for h_{MN} and the strong coupling limit in which $l \to \infty$.

For the free theory, the gauge symmetry is

$$\delta h_{MN} = \partial_{(M} \xi_{N)} \tag{3.3}$$

and the invariant field strength is the linearised Riemann tensor

$$R_{MN\,PQ} = \partial_M \partial_P h_{NQ} + \ldots = -4 \partial_{[M} h_{N][P,Q]} \tag{3.4}$$

This satisfies

$$R_{MN\,PQ} = R_{PQ\,MN} \tag{3.5}$$

together with the first Bianchi identity

$$R_{[MN\,P]Q} = 0 \tag{3.6}$$

and the second Bianchi identity

$$\partial_{[S}R_{MN]\,PQ}=0 \tag{3.7}$$

The natural free field equation in $D\geq 4$ is the linearised Einstein equation

$$R^{P}{}_{M\,PN}=0 \tag{3.8}$$

which then implies

$$\partial^{M}R_{MN\,PQ}=0 \tag{3.9}$$

where indices are raised and lowered with a flat background metric η_{MN}. In $D=3$, the Weyl tensor vanishes identically so that the Riemann tensor is completely determined by the Ricci tensor, and the field equation (3.8) implies that the field strength (3.4) vanishes and h_{MN} is pure gauge. The simplest non-trivial linear field equation is in $D=3$ is

$$R^{MN}{}_{MN}=0 \tag{3.10}$$

representing one degree of freedom. Indeed, in $D=3$, a free graviton satisfying the field equation (3.10) can be dualised to a free scalar ϕ by the relation

$$R^{MN\,PQ}=\epsilon^{MNR}\epsilon^{PQS}\partial_{R}\partial_{S}\phi \tag{3.11}$$

with (3.10) implying $\Box\phi=0$. In $D=2$, the Riemann tensor is completely determined by the Ricci scalar, so that the field equation (3.10) only has trivial solutions and there is no non-trivial linear field equation.

In four Euclidean dimensions, one can consistently impose the self-duality condition

$$R_{MN\,PQ}=\frac{1}{2}\epsilon_{MNST}R^{ST}{}_{PQ} \tag{3.12}$$

which implies the field equations (3.8), but is stronger.

3.2. The 4th Rank Tensor Gauge Field

Consider a gauge field $C_{MN\,PQ}$ in D dimensions with the algebraic properties of the Riemann tensor

$$C_{MN\,PQ} = -C_{NM\,PQ} = -C_{MN\,QP} = C_{PQ\,MN}, \qquad C_{[MNP]Q} = 0 \tag{3.13}$$

and the gauge symmetry

$$\delta C_{MN\,PQ} = \partial_{[M}\chi_{N]PQ} + \partial_{[P}\chi_{Q]MN} - 2\partial_{[M}\chi_{NPQ]} \tag{3.14}$$

with parameter $\chi_{MPQ} = -\chi_{MQP}$. Such gauge fields were considered in four dimensions in [5] and arise in the (4,0) supermultiplet in $D = 6$ [1]. The invariant field strength is

$$G_{MNP\,QRS} = \partial_M \partial_Q C_{NP\,RS} + \ldots = 9\partial_{[M}C_{NP]\,[QR,S]} \tag{3.15}$$

so that

$$G_{MNP\,QRS} = G_{[MNP]\,[QRS]} = G_{QRS\,MNP} \tag{3.16}$$

This satisfies the first Bianchi identity

$$G_{[MNP\,Q]RS} = 0 \tag{3.17}$$

and the second Bianchi identity

$$\partial_{[T}G_{MNP]\,QRS} = 0 \tag{3.18}$$

The natural linear free field equation in $D \geq 6$ is

$$G^{Q}{}_{NP\,QRS} = 0 \tag{3.19}$$

which then implies (using (3.16),(3.17))

$$\partial^M G_{MNP\,QRS} = 0 \tag{3.20}$$

In $D \leq 5$, the trace-free part of the field strength vanishes identically, so that

in $D = 5$

$$G_{\mu\nu\rho}{}^{\sigma\tau\lambda} = 9G_{[\mu\nu}{}^{[\sigma\tau}\delta_{\rho]}{}^{\lambda]} - 9G_{[\mu}{}^{[\sigma}\delta_{\nu\rho]}{}^{\tau\lambda]} + G\delta_{\mu\nu\rho}{}^{\sigma\tau\lambda} \tag{3.21}$$

where

$$G_{\mu\nu}{}^{\sigma\tau} = G_{\mu\nu\rho}{}^{\sigma\tau\rho}, \qquad G_{\mu\sigma} = G_{\mu\nu\sigma}{}^{\nu}, \qquad G = G_{\mu}{}^{\mu} \tag{3.22}$$

(here $\mu, \nu = 0, 1, \ldots 4$). This is analogous to the fact that the trace-free part of the Riemann tensor, the Weyl tensor, vanishes in $D \leq 3$. The field equation for $C_{\mu\nu\,\sigma\tau}$ given by

$$G_{\mu\nu\rho\,\sigma\tau}{}^{\rho} = 0 \tag{3.23}$$

then implies that the field strength vanishes, $G_{\mu\nu\rho\,\sigma\tau\lambda} = 0$, in dimensions $D \leq 5$, so that $C_{\mu\nu\,\rho\sigma}$ is pure gauge. Thus the field equation (3.23) with one contraction is trivial and the simplest non-trivial linear 2nd order field equation is

$$G^{\sigma\tau}{}_{\rho\,\sigma\tau\lambda} = 0 \tag{3.24}$$

with two contractions. A gauge field $C_{\mu\nu\,\rho\sigma}$ satisfying this field equation in $D = 5$ represents 5 degrees of freedom and can be dualised to a linearised graviton [1] $\hat{h}_{\mu\nu}$ with linearised curvature $\hat{R}_{\mu\nu\,\rho\sigma}(\hat{h})$ given by

$$\hat{R}_{\mu\nu\,\rho\sigma} = \frac{1}{36}\epsilon_{\mu\nu\alpha\beta\gamma}G^{\alpha\beta\gamma\,\kappa\lambda\tau}\epsilon_{\rho\sigma\kappa\lambda\tau} \tag{3.25}$$

or $\hat{R} = *G*$. The Bianchi identities and field equation (3.24) for $C_{\mu\nu\,\rho\sigma}$ then imply the Bianchi identities and linearised Einstein equation for $\hat{R}$.

A natural 2nd order field equation in $D = 4$ for C_{mnpq} $(m, n = 0, 1, 2, 3)$ is

$$G^{mnp}{}_{mnp} = 0 \tag{3.26}$$

(The field equations given by requiring single or double contractions of G to vanish are trivial in $D \leq 4$ and force C to be pure gauge. The simplest equation with

non-trivial solutions is (3.26) with triple contractions of G.) If it satisfies this, it can be dualised to a scalar field $\hat{\phi}$ defined by

$$\partial_m \partial_n \hat{\phi} = (*G*)_{mn} \tag{3.27}$$

In $D < 4$, there is no non-trivial linear field equation for such gauge fields.

In [5], a $D = 4$ gauge field C_{mnpq} with the same algebraic properties (3.13) was considered, and there it was shown that an unusual higher derivative field equation for C_{mnpq} gave a system that was dual to a Maxwell vector field. The name notivarg was proposed for C_{mnpq} in [5], in analogy with the name notoph proposed in [6] for a 2-form gauge field B_{mn} in four dimensions. Here, the 2nd order field equation (3.19) implies that C_{mnpq} is dual to a scalar, not a vector field.

The gauge field $C_{MN\,PQ}$ in $D = 6$ with field equation (3.19) represents 10 degrees of freedom in the $(5,1) + (1,5)$ representation of the little group $SU(2) \times SU(2)$ [1]. In $5+1$ dimensions, one can consistently impose the the self-duality constraint

$$G = *G \tag{3.28}$$

which then implies $G = G* = *G*$ where

$$\begin{aligned} (*G)_{MNP\,QRS} &= \frac{1}{6} \epsilon_{MNPTUV} G^{TUV}{}_{QRS} \\ (G*)_{MNP\,QRS} &= \frac{1}{6} \epsilon_{QRSTUV} G_{MNP}{}^{TUV} \end{aligned} \tag{3.29}$$

This halves the degrees of freedom to 5 in the $(5,1)$ representation of the little group. The 4-th rank gauge field in the the (4,0) multiplet satisfies such a self-duality constraint [1]. The self-duality constraint (3.28) and the Bianchi identities (3.17),(3.18) imply the field equations (3.19),(3.20).

3.3. Reduction from Six to Five Dimensions

Reducing a general unconstrained 4th rank $D = 6$ gauge field $C_{MN\,PQ}$ to $D = 5$ on a circle of radius r gives the fields

$$h_{\mu\nu} = C_{\mu 5\,\nu 5}, \qquad D_{\mu\nu\,\rho} = C_{\mu\nu\,\rho 5}, \qquad C_{\mu\nu\,\rho\sigma} \tag{3.30}$$

with the algebraic properties

$$h_{[\mu\nu]} = 0, \qquad D_{[\mu\nu\,\rho]} = 0, \qquad C_{[\mu\nu\,\rho]\sigma} = 0 \tag{3.31}$$

The linear field strengths $R_{\mu\nu\,\rho\sigma}$ for $h_{\mu\nu}$, $S_{\mu\nu\rho\,\sigma\tau}$ for $D_{\mu\nu\rho}$ and $G_{\mu\nu\rho\,\sigma\tau\lambda}$ for $C_{\mu\nu\,\rho\sigma}$ are given by

$$\begin{aligned} R_{\mu\nu\,\rho\sigma} &= G_{\mu\nu 5\,\rho\sigma 5} \\ S_{\mu\nu\rho\,\sigma\tau} &= G_{\mu\nu\rho\,\sigma\tau 5} \\ &G_{\mu\nu\rho\,\sigma\tau\lambda} \end{aligned} \tag{3.32}$$

so that

$$S_{\mu\nu\rho\,\sigma\tau} = -6\partial_{[\mu} D_{\nu\rho]\,[\sigma,\tau]}, \qquad G_{\mu\nu\rho\,\sigma\tau\lambda} = 9\partial_{[\mu} C_{\nu\rho]\,[\sigma\tau,\lambda]} \tag{3.33}$$

and $R_{\mu\nu\,\rho\sigma}$ is the linearised Riemann tensor for $h_{\mu\nu}$, given by (3.4).

Consider first the natural free field equations in $D = 5$ for the gauge fields $h_{\mu\nu}, D_{\mu\nu\,\rho}, C_{\mu\nu\,\rho\sigma}$ that are given by

$$\begin{aligned} R^{\rho}{}_{\nu\,\rho\sigma} &= 0 \\ S^{\sigma}{}_{\nu\rho\,\sigma\tau} &= 0 \\ G^{\sigma\tau}{}_{\rho\,\sigma\tau\lambda} &= 0 \end{aligned} \tag{3.34}$$

(As discussed above, the field equation for C given by $G_{\mu\nu\rho\,\sigma\tau}{}^{\rho} = 0$ that might have been expected by comparison to the $D = 6$ equation (3.19) would be trivial, so (3.24) is the simplest non-trivial possibility.) If the field equations (3.34) are

satisfied, then each of the gauge fields represents 5 degrees of freedom. Indeed, it was seen in [1] that in the linearised $D = 5$ theory a linearised graviton $h_{\mu\nu}$ has dual representations in terms of fields $D_{\mu\nu\,\rho}$ or $C_{\mu\nu\,\rho\sigma}$, so $C_{\mu\nu\,\rho\sigma}$ can be dualised to give a 2nd graviton (i.e. free symmetric tensor gauge field) $\hat{h}_{\mu\nu}$ and $D_{\mu\nu\,\rho}$ can be dualised to a third one $\tilde{h}_{\mu\nu}$. Linearised gravitons $\tilde{h}_{\mu\nu}$ and $\hat{h}_{\mu\nu}$ which are dual to $D_{\mu\nu\rho}$ and $C_{\mu\nu\,\rho\sigma}$ respectively are introduced by requiring that the corresponding linearised curvatures $\tilde{R}_{\mu\nu\,\rho\sigma}(\tilde{h})$ and $\hat{R}_{\mu\nu\,\rho\sigma}(\hat{h})$ satisfy

$$\begin{aligned}\tilde{R}_{\mu\nu\,\rho\sigma} &= r\frac{1}{6}\epsilon_{\mu\nu\alpha\beta\gamma}S^{\alpha\beta\gamma}{}_{\rho\sigma}\\ \hat{R}_{\mu\nu\,\rho\sigma} &= r^2\frac{1}{36}\epsilon_{\mu\nu\alpha\beta\gamma}G^{\alpha\beta\gamma\,\kappa\lambda\tau}\epsilon_{\rho\sigma\kappa\lambda\tau}\end{aligned} \tag{3.35}$$

or

$$\tilde{R} = r * S, \qquad \hat{R} = r^2 * G* \tag{3.36}$$

provided the field equations are those in (3.34). Then the Bianchi identities and field equations of each field imply those of its dual.

If the reduction to $D = 5$ of a $D = 6$ gauge field satisfying (3.19) gave the three gauge fields h, D, C with the field equations (3.34), there would be 15 degrees of freedom and the resulting system could be dualised to one with three free gravitons $h, \tilde{h}, \hat{h}$. However, there were only 10 degrees of freedom in $D = 6$, so the dynamics here must be different. In fact, the reduction of the $D = 6$ gauge field gives stronger field equations than (3.34) and only two independent gravitons remain, as will now be shown.

The reduction of the $D = 6$ field equation (3.19) gives the first two field equations of (3.34), but the field equation for C is

$$G_{\mu\nu\rho\,\sigma\tau}{}^{\rho} = -\frac{1}{r^2}R_{\mu\nu\,\sigma\tau} \tag{3.37}$$

This implies the 'natural' field equation $G^{\sigma\tau}{}_{\rho\sigma\tau\lambda} = 0$ of (3.34) (using the first field equation $R_{\mu\nu} = 0$), but is stronger. Since (3.37) fixes the trace of $G_{\mu\nu\rho\,\sigma\tau\lambda}$, it

completely determines the whole of $G_{\mu\nu\rho\,\sigma\tau\lambda}$ in terms of $R_{\mu\nu\,\sigma\tau}$:

$$G_{\mu\nu\rho}{}^{\alpha\beta\gamma} = -9\frac{1}{r^2}R_{[\mu\nu}{}^{[\alpha\beta}\delta_{\rho]}{}^{\gamma]} \tag{3.38}$$

(using $R_{\mu\nu} = 0$). This implies that C is not an independent degree of freedom, and can be solved for in terms of h as

$$C_{\mu\nu}{}^{\rho\sigma} = -4\frac{1}{r^2}h_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]} \tag{3.39}$$

up to gauge transformations, or equivalently after the field redefinition

$$C_{\mu\nu}{}^{\rho\sigma} \to \bar{C}_{\mu\nu}{}^{\rho\sigma} = C_{\mu\nu}{}^{\rho\sigma} + 4\frac{1}{r^2}h_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]} \tag{3.40}$$

the field $\bar{C}$ has the field equation

$$\bar{G}_{\mu\nu\rho\,\sigma\tau}{}^{\rho} = 0 \tag{3.41}$$

implying that $\bar{C}$ is pure gauge and so trivial. In terms of the dual variables, this implies that the graviton $\hat{h}_{\mu\nu}$ (dual to C) satisfies $\hat{R}_{\mu\nu\sigma\rho} = R_{\mu\nu\sigma\rho}$ so that (up to gauge transformations) $\hat{h}_{\mu\nu} = h_{\mu\nu}$. Then the system has only 10 independent degrees of freedom, represented by the 5 of the graviton $h_{\mu\nu}$ and the 5 of $\tilde{h}_{\mu\nu}$ or its dual $D_{\mu\nu\,\rho}$.

Suppose now that the $D = 6$ gauge field is self-dual, satisfying (3.28). This gives the $D = 5$ constraints

$$R = r * S = r^2 * G* \tag{3.42}$$

which implies that C, D are not independent degrees of freedom but can be eliminated in terms of h. In terms of the dual variables, this implies that (up to gauge transformations)

$$\hat{h}_{\mu\nu} = \tilde{h}_{\mu\nu} = h_{\mu\nu} \tag{3.43}$$

so that there is only one independent graviton and only 5 degrees of freedom, as required.

3.4. Reduction from Six to Four Dimensions

Reducing the gauge field C_{MNPQ} to $D=4$ on a 2-torus with constant metric γ_{ij} gives the gauge fields and their corresponding field strengths

$$\begin{aligned}
&C_{mnpq}, \qquad && G_{mnpqrs} = \partial_m \partial_q C_{nprs} + \dots \\
&D_{mnp\,i} = C_{mnpi}, \qquad && S_{mnpqr\,i} = \partial_m \partial_q D_{npr\,i} + \dots \\
&B_{mn} = \frac{1}{2}\epsilon^{ij} C_{mnij}, \qquad && H = dB \\
&h_{mn\,ij} = -C_{m(ij)n}, \qquad && R_{mnpq\,ij} = \partial_m \partial_p h_{nq\,ij} + \dots \\
&A_{mi} = \frac{1}{2}\epsilon^{jk} C_{mijk}, \qquad && F_i = dA_i \\
&\phi = \frac{1}{4}\epsilon^{ij}\epsilon^{kl} C_{ij\,kl}, \qquad && P = d\phi
\end{aligned} \tag{3.44}$$

where, as in section 2, ϵ_{ij} has components $\epsilon_{12} = \sqrt{det \gamma_{ij}}$ and the indices i,j are raised and lowered with the metric γ_{ij}. Dimensionally reducing the field strength G_{MNPQRS} gives

$$\begin{aligned}
&G_{mnp\,qrs} \\
&G_{mnp\,qri} = S_{mnpqr\,i} \\
&G_{mnp\,qij} = \partial_q H_{mnp} \epsilon_{ij} \\
&G_{mn(ij)pq} = R_{mnpq\,ij} \\
&G_{mni\,pjk} = \partial_p F_{mni} \epsilon_{jk} \\
&G_{mij\,nkl} = \epsilon_{ij}\epsilon_{kl} \partial_m \partial_n \phi
\end{aligned} \tag{3.45}$$

The usual free field equations for the 0,1,2 form gauge fields ϕ, A_i, B are

$$d * P = 0, \qquad d * F = 0, \qquad d * H = 0 \tag{3.46}$$

and the 2-form B can be dualised to a scalar $\tilde{\phi}$ defined by

$$d\tilde{\phi} = *H \tag{3.47}$$

The natural free 2nd order field equations for the three symmetric tensor gauge

fields $h_{mn\,ij} = h_{(mn)\,(ij)}$ are the linearised Einstein equations

$$R^p{}_{npq\,ij} = 0 \tag{3.48}$$

As above, the natural 2nd order field equation in $D = 4$ for C_{mnpq} is

$$G^{mnp}{}_{mnp} = 0 \tag{3.49}$$

which implies it can be dualised to a scalar field $\hat{\phi}$ defined by

$$\partial_m \partial_n \hat{\phi} = (*G*)_{mn} \tag{3.50}$$

Similarly, the natural 2nd order field equation in $D = 4$ for $D_{mnp\,i}$ is

$$S^{mn}{}_{pmn\,i} = 0 \tag{3.51}$$

and if it satisfies this, $D_{mnp\,i}$ can be dualised to vector fields $\tilde{A}_{mi}$ through the relation

$$S_{mnp\,qr\,i} = \epsilon_{mnpt} \partial^t \tilde{F}_{qr\,i} \tag{3.52}$$

Reducing the $D = 6$ field equation (3.19) gives the equations (3.46) but the equations for the remaining gauge fields are different from (3.48),(3.49),(3.51). The field equation for $D_{mnp\,i}$ obtained by reduction is

$$S^m{}_{np\,mq\,i} = -\epsilon_{ij} \partial_q F_{np}{}^j \tag{3.53}$$

This implies (3.51), but is stronger. The trace-free part of S_{mnpqri} vanishes identically in $D = 4$, so that (3.53) determines the field strength S_{mnpqri} completely in

terms of A_i, and implies that

$$D_{mn\,pi} \propto \epsilon_{ij}\eta_{p[m}A_{n]}{}^j \tag{3.54}$$

up to gauge transformations, so that D does not represent independent degrees of freedom. In terms of the dual potentials $\tilde{A}_i$, the constraint implies

$$\tilde{F}_i = \epsilon_{ij}(*F)^j \tag{3.55}$$

so that the potentials $\tilde{A}_i$ are dual to $\epsilon_{ij}A^j$.

The remaining field equations are

$$R^p{}_{npq\,ij} = -\gamma_{ij}\partial_n\partial_q\phi \tag{3.56}$$

and

$$G^r{}_{mn\,rpq} = -\gamma^{ij}R_{mnpq\,ij} \tag{3.57}$$

and these imply (3.48),(3.49), but are stronger. It will be useful to decompose the three gravitons as

$$h_{mn\,ij} = g_{mn}\gamma_{ij} + h'{}_{mn\,ij} \tag{3.58}$$

where $\gamma^{ij}h'{}_{mnij} = 0$. As in $D = 5$, (3.57) determines C_{mnpq} completely in terms of the graviton $g_{mn} \equiv \frac{1}{2}\gamma^{ij}h_{mnij}$. Furthermore, the equations (3.56),(3.57) determine the curvature of g_{mn}, defined by

$$R_{mnpq} = \frac{1}{2}\gamma^{ij}R_{mnpqij} \tag{3.59}$$

completely in terms of ϕ. Then C and g_{mn} are constrained to be

$$\begin{aligned} C_{mnpq} &= \frac{2}{3}\eta_{m[p}\eta_{q]n}\phi \\ g_{mn} &= -\frac{1}{2}\eta_{mn}\phi \end{aligned} \tag{3.60}$$

up to gauge transformations. In terms of the scalar field $\hat{\phi}$ dual to C_{mnpq}, the field

equations imply

$$\hat{\phi} = \phi \tag{3.61}$$

To summarise, on dimensionally reducing the gauge field C_{MNPQ} satisfying (3.19), there are three scalars $\phi, \tilde{\phi}, \hat{\phi}$ satisfying one constraint (3.61) leaving two independent scalars, there are four vector fields $A_i, \tilde{A}_i$ satisfying two constraints (3.55) leaving two independent vector fields and there are three symmetric tensor gauge fields $h_{mn\,ij}$ satisfying one constraint given by (3.57) or (3.60), leaving two independent ones. Then the independent degrees of freedom can be taken to be represented by the two scalars $\phi, \tilde{\phi}$, the two vector fields A_i and the two gravitons $h'_{mn\,ij}$, which are a symmetric traceless tensor in the i, j indices, giving a total of 10 degrees of freedom, as required. The field equations are manifestly invariant under $SL(2, \mathbb{R})$, with the two vector fields A_i transforming as a doublet and the gravitons $h_{mn\,ij}$ transforming as a symmetric tensor, subject to the $SL(2, \mathbb{R})$-invariant constraint that they be trace-free, $\gamma^{ij} h_{mn\,ij} = 0$.

The $D = 6$ self-duality condition (3.28) halves the number of degrees of freedom, leaving in $D = 4$ one scalar, one vector field and one graviton. The constraint (3.28) implies, on reduction, that

$$\begin{aligned}
&G_{mnpqrs} = \epsilon_{mnpt}\epsilon_{pqru}\partial^t\partial^u\phi \\
&S_{mnpqr\,i} = \epsilon_{mnps}\partial^s F_{qr\,i} \\
&\partial_q H_{mnp} = \epsilon_{mnps}\partial^s\partial_q\phi \\
&R_{mnpq\,ij} = (*R)_{mnpq\,k(j}{J_{i)}}^k - \gamma_{ij}\epsilon_{mnrs}\partial^r {H^s}_{pq} \\
&\partial_p F_{mni} = {J_i}^j\partial_p(*F)_{mnj}
\end{aligned} \tag{3.62}$$

As before, these determine C_{mnpq} or $\hat{\phi}$ in terms of ϕ (3.60),(3.61) and $D_{mnp\,i}$ or $\tilde{A}_i$ in terms of A_i (3.54),(3.55). In addition, they imply

$$H = *d\phi \tag{3.63}$$

(up to the addition of a constant 3-form which could arise as a constant of inte-

gration) so that B is dual to ϕ, or equivalently

$$\phi = \tilde{\phi} \tag{3.64}$$

so that now $\phi = \tilde{\phi} = \hat{\phi}$ and only one of the three scalars is independent. Furthermore, the two 1-form gauge fields satisfy the duality constraint

$$F_{mni} = J_i{}^j (*F)_{mnj} \tag{3.65}$$

(again suppressing a possible constant 2-form) so that A_2 is dual to A_1, and only one of the 1-form gauge fields is independent.

The constraint on the curvature tensors again determines g_{mn} in terms of ϕ through (3.60) (using (3.63)). Equivalently, the field redefinition

$$h_{mn\,ij} \to \bar{h}_{mn\,ij} = h_{mn\,ij} + \gamma_{ij}\eta_{mn}\phi \tag{3.66}$$

brings the constraint to the form

$$\bar{R}_{mnpq\,ij} = J_i{}^k (*\bar{R})_{mnpq\,kj} \tag{3.67}$$

or

$$\bar{R} = J * \bar{R} \tag{3.68}$$

This implies that

$$\gamma^{ij}\bar{R}_{mnpq\,ij} = 0 \tag{3.69}$$

so that (after the redefinition (3.66)) $\bar{g}_{mn} = \frac{1}{2}\gamma^{ij}\bar{h}_{mn\,ij}$ is trivial and can be gauged to zero. This also implies that the right hand side of (3.67) is symmetric in the

indices i, j and that

$$\bar{R}_{mnpq\,ij} = J_i{}^k (\bar{R}*)_{mnpq\,kj} \tag{3.70}$$

The 1st Bianchi identities

$$\bar{R}_{[mnp]q\,ij} = 0 \tag{3.71}$$

and the self-duality condition (3.67) then imply the field equations

$$\eta^{mp} \bar{R}_{mnpq\,ij} = 0 \tag{3.72}$$

Moreover, (3.67) gives a duality relation between the two remaining gravitons, so that only one independent graviton remains. Then the $D = 6$ self-duality condition (3.28) indeed halves the number of degrees of freedom, leaving one scalar, one vector field and one graviton in $D = 4$ (which is of course what is to be expected from the reduction of the $D = 5$ graviton $h_{\mu\nu}$). However, the self-duality conditions (3.67),(3.69) are $SL(2,\mathbb{R})$ invariant, so that the classical $D = 4$ theory has an $SL(2,\mathbb{R})$ symmetry.

4. Gravitational S-Duality in Four Dimensions

In this section, the $SL(2,\mathbb{Z})$ duality will be discussed both for the general systems obtained by reducing (2.3) or (3.19), and for the self-dual systems satisfying (2.19) or (3.67). The 2-torus metric can be written in terms of a complex modulus

$$\tau = \tau_1 + i\tau_2 = \theta + \frac{i}{g^2} \tag{4.1}$$

and the volume V as

$$\gamma_{ij} = \frac{V}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \tag{4.2}$$

The $SL(2)$ invariant action (2.13) for the doublet (F_1, F_2) of independent field

strengths is

$$S=\int d^4x\frac{1}{\tau_2}\left(|\tau|^2(F_1)^2+(F_2)^2-2\tau_1F_1\cdot F_2\right) \tag{4.3}$$

and reduces to

$$S=\int d^4x\left(\frac{1}{g^2}(F_1)^2+g^2(F_2)^2\right) \tag{4.4}$$

for rectangular tori with $\theta=0$. The self-duality equation for F_i (2.19) (which implies the vanishing of the action (4.3)) gives

$$F_2=\frac{1}{g^2}*F_1+\theta F_1 \tag{4.5}$$

It will be convenient to parameterise the triplets of metric and curvature tensors (after the redefinition (3.66)) as

$$\bar{h}_{mn\,ij}=\begin{pmatrix} h_{mn} & \tilde{h}_{mn} \\ \tilde{h}_{mn} & \hat{h}_{mn}\end{pmatrix}, \qquad \bar{R}_{mnpq\,ij}=\begin{pmatrix} R_{mnpq} & \tilde{R}_{mnpq} \\ \tilde{R}_{mnpq} & \hat{R}_{mnpq}\end{pmatrix} \tag{4.6}$$

Then (3.69) implies

$$\hat{R}_{mnpq}-2\tau_1\tilde{R}_{mnpq}+|\tau|^2R_{mnpq}=0 \tag{4.7}$$

so that

$$\hat{h}_{mn}-2\tau_1\tilde{h}_{mn}+|\tau|^2h_{mn}=0 \tag{4.8}$$

up to gauge transformations.

It is straightforward to find the quadratic action implying the field equation (3.19). In physical gauge, with transverse traceless C_{MNPQ} [1] the action simplifies

to

$$\int d^6x\sqrt{-g}\; C_{MNPQ}\Box C^{MNPQ} \tag{4.9}$$

Reducing on a 2-torus, this gives the following physical gauge action for the gravitons $\bar{h}_{mn\,ij}$, which are also transverse and traceless:

$$S = \frac{1}{4V}\int d^4x\, \tilde{\gamma}^{ik}\tilde{\gamma}^{jl}\bar{h}_{mnij}\Box\bar{h}^{mn}{}_{kl} \tag{4.10}$$

which is manifestly $SL(2)$ invariant. If $\theta = 0$ this action is

$$S = \frac{1}{4V}\int d^4x \left(\frac{1}{g^4}h_{mn}\Box h^{mn} + 2\tilde{h}_{mn}\Box\tilde{h}^{mn} + g^4\hat{h}_{mn}\Box\hat{h}^{mn}\right) \tag{4.11}$$

while the constraint (3.69) becomes

$$\hat{h} = -\frac{1}{g^4}h \tag{4.12}$$

and imposing this reduces the action (4.11) for the two gravitons $h, \tilde{h}$ to

$$S = \frac{1}{2V}\int d^4x \left(\frac{1}{g^4}h_{mn}\Box h^{mn} + \tilde{h}_{mn}\Box\tilde{h}^{mn}\right) \tag{4.13}$$

The covariant form of the action (4.11) is the sum of the linearised Einstein-Hilbert actions for $h, \tilde{h}, \hat{h}$.

The duality constraint (3.67) gives the following relations between the curvature tensors (suppressing the indices)

$$\begin{aligned} R &= \frac{1}{\tau_2}(-*\tilde{R} + \tau_1 * R) \\ \tilde{R} &= \frac{1}{\tau_2}(-*\hat{R} + \tau_1 * \tilde{R}) \\ \hat{R} &= \frac{1}{\tau_2}(|\tau|^2 * \tilde{R} - \tau_1 * \hat{R}) \end{aligned} \tag{4.14}$$

This implies that $\tilde{R}, \hat{R}$ are given in terms of R by

$$\begin{aligned}\tilde{R} &= \frac{1}{g^2} * R - \theta R \\ \hat{R} &= 2\tau_1\tau_2 * R - (\tau_1^2 - \tau_2^2) R\end{aligned} \tag{4.15}$$

The one remaining independent graviton can be taken to be h with $\theta = 0$ action

$$S = \frac{1}{2l^2} \int d^4x \, h_{mn} \Box h^{mn} \tag{4.16}$$

where the Planck length is given by

$$l = \sqrt{V} g^2 \tag{4.17}$$

The reduction gives two dimensionless coupling constants, g, θ. While g can be absorbed into the the gravitational coupling l, there is the interesting possibility of a gravitational θ-parameter.

Under the action of $SL(2)$, $\bar{R}_{mnpq\, ij}$ transforms as a symmetric tensor

$$\bar{R}_{mnpq} \to S \bar{R}_{mnpq} S^t \tag{4.18}$$

with the invariant tracefree condition (3.69), F_i transforms as a vector $F \to SF$ and τ transforms through a fractional linear transformation. For F_i satisfying the self-dual condition (2.19), an $SL(2)$ transformation takes F_1 to a linear combination of F_1 and $*F_1$ while for curvatures satisfying the self-duality condition (3.67) it takes R to a linear combination of R and $*R$ and mixes the field equations

$$\eta^{mp} R_{mnpq} = 0 \tag{4.19}$$

with the Bianchi identities

$$\eta^{mp} (*R)_{mnpq} = 0 \tag{4.20}$$

The action of the $SL(2,\mathbb{Z})$ element

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.21}$$

is to take

$$\tau \to -\frac{1}{\tau} \tag{4.22}$$

and

$$F_1 \to F_2, \qquad F_2 \to -F_1 \tag{4.23}$$

while

$$R \to \hat{R}, \qquad \hat{R} \to R, \qquad \tilde{R} \to -\tilde{R} \tag{4.24}$$

Note that this preserves the constraint (4.7). For self-dual F_i, the transformation is the standard duality transformation

$$F_1 \to \frac{1}{g^2} * F_1 + \theta F_1 \tag{4.25}$$

while for self-dual $\bar{R}$

$$R_{mnpq} \to 2\tau_1\tau_2(*R)_{mnpq} - (\tau_1^2 - \tau_2^2)R_{mnpq} \tag{4.26}$$

For $\theta = 0$, this $SL(2,\mathbb{Z})$ transformation takes $g \to 1/g$ and so relates strong and weak coupling regimes. Both the linear gravity and Maxwell theory are self-dual, with the strong coupling regime described by the same theory.

For Maxwell theory, maintaining duality in the presence of sources requires introducing a magnetic current $\tilde{J}$ as well as an electric current J with

$$dF = *\tilde{J}, \qquad d * F = *J \tag{4.27}$$

and $(J, \tilde{J})$ forming an $SL(2)$ doublet. In regions in which $\tilde{J}$ vanishes, F can be solved for in terms of a potential A, $F = dA$, while in regions in which J vanishes,

$*F$ can be solved for in terms of a dual potential $\tilde{A}$, $*F = d\tilde{A}$. Similarly, duality requires introducing energy-momentum tensors T_{mnij} for all three gravitons h_{mnij} in the general system (4.11), while for the self-dual system satisfying (3.67), this requires introducing a 'magnetic' energy-momentum tensor $\tilde{T}_{mn}$ as well as the usual T_{mn} with

$$\eta^{mp} R_{mnpq} = T_{nq} + \frac{1}{2}\eta_{nq} T \tag{4.28}$$

and a source for the first Bianchi identity

$$\eta^{mp} (*R)_{mnpq} = \tilde{T}_{nq} + \frac{1}{2}\eta_{nq} \tilde{T} \tag{4.29}$$

In regions in which $\tilde{T}_{mn}$ vanishes, R_{mnpq} can be solved for in terms of a graviton h_{mn}, while in regions in which T_{mn} vanishes, $\tilde{R}_{mnpq}$ can be solved for in terms of a dual graviton $\tilde{h}_{mn}$.

5. Symmetries of the Linear (4,0) Theory and its Compactifications

The maximal supergravity theory in D dimensions has a rigid duality symmetry G which is broken to a discrete U-duality subgroup $G(\mathbb{Z})$ in the quantum theory; for $D = 4$, $G = E_{7(+7)}$ [11] and for $D = 5$, $G = E_{6(+6)}$ [12]. The scalar fields take values in the coset space G/H where H is the maximal compact subgroup of G, and the theory can be formulated with local H symmetry [11,10]. However, the free limit of these supergravities in general have much larger symmetry groups and the symmetry groups of the linearised theories will play a role here.

The scalar fields in a supergravity theory take values in some target space $\mathcal{M}$. The non-linear sigma-model action has an invariance under the diffeomorphism group of $\mathcal{M}$, the group $Diff(\mathcal{M})$ of pseudo-symmetries or sigma-model symmetries. The isometry subgroup of this $Isom(\mathcal{M})$ are proper symmetries of the action. In maximal supergravity theories, $\mathcal{M}$ is a coset space G/H and the isometry group is G, giving the rigid G duality symmetry. For n free scalar fields, the target

space is n-dimensional flat space and the isometry group is the Euclidean group $IO(n) = O(n) \ltimes \mathbb{R}^n$ of translations, reflections and rotations. The free scalar field equations have the larger symmetry $IGL(n) = GL(n) \ltimes \mathbb{R}^n$. The $N = 8, D = 5$ supergravity has 42 scalar fields in $G/H = E_6/USp(8)$ and in the free limit the symmetry E_6 of the scalar kinetic term is increased to $IGL(42)$, which contains the group contraction of E_6 given by $USp(8) \ltimes \mathbb{R}^{42}$ as a subgroup. In the full theory, the scalar self-interactions break the symmetries not in this subgroup and modify the translation symmetries so that $USp(8) \ltimes \mathbb{R}^{42}$ becomes E_6. The situation is similar for $N = 8, D = 4$ supergravity which has 70 scalar fields in $G/H = E_7/SU(8)$ and in the free limit the symmetry E_7 is increased to $IGL(70)$, which contains the group contraction of E_7 given by $SU(8) \ltimes \mathbb{C}^{35}$ as a subgroup.

The $D = 5, N = 8$ supergravity has 27 vector gauge fields and so the classical free vector field equations have a $GL(27, \mathbb{R})$ symmetry, broken to $GL(27, \mathbb{Z})$ in the quantum theory. The interactions with the scalars break the $GL(27)$ symmetry to E_6 with the vector fields transforming as a **27**. The $D = 4, N = 8$ supergravity has 28 vector gauge fields with field strengths F_A, $A = 1, ..., 28$, satisfying $dF_A = 0$. The vector field equations can be written as $dG_A = 0$ where the 28 dual field strengths G_A are given by $G_A = *F_A$ in the free theory, while in the interacting theory $G_A = *(\delta S/\delta F_A) = *F_A + \ldots$. The field equations imply that the G_A can be written in terms of dual potentials $\tilde{A}_A$, $G_A = d\tilde{A}_A$. The 56-vector of field strengths given by

$$\mathcal{F} = \begin{pmatrix} \hat{F}_A \\ \hat{G}_A \end{pmatrix} \tag{5.1}$$

then satisfies (setting the fermions to zero) [11]

$$\mathcal{F} = J * \mathcal{F}, \qquad d\mathcal{F} = 0 \tag{5.2}$$

where J is the 56×56 matrix

$$J = \mathcal{V}^{-1} \Omega \mathcal{V}, \qquad \Omega \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.3}$$

and $\mathcal{V}(\phi)$ is the scalar-dependent 56-bein parameterising the scalar coset space. In the free case, $J = \Omega$ and the equations (5.2) have an $Sp(56, \mathbb{R})$ symmetry (the 56×56 matrices preserving Ω) [8] which is broken to $Sp(56, \mathbb{Z})$ in the quantum theory. In the full theory the interactions with the scalar fields break the $Sp(56)$ to E_7 and these equations are supercovariantized with fermion bilinears [11].

Finally, the linearised Einstein equations in $D = 4$ can be expressed in terms of the matrix of dual gravitons as (3.67) with a manifest $SL(2)$ symmetry, although this is not a symmetry of the non-linear Einstein equations. Thus the classical free bosonic field equations of linearised $N = 8, D = 4$ supergravity have a symmetry

$$SL(2) \times Sp(56) \times IGL(70) \tag{5.4}$$

while those of linearised $N = 8, D = 5$ supergravity have a symmetry

$$GL(27) \times IGL(42) \tag{5.5}$$

The bosonic field equations of the free (4,0) theory

$$G = *G, \qquad H_a = *H_a, \qquad \Box \phi^i = 0 \tag{5.6}$$

also have the symmetry (5.5), with the 42 scalars ϕ^i taking values in $\mathbb{R}^{42}$ and acted on by $IGL(42)$ while the 27 self-dual 3-form field strengths H_a transform as a **27** of $GL(27)$.

The dimensional reduction of the free (4,0) theory on a 2-torus must have the symmetry (5.5) together with an extra $SL(2)$ symmetry from the reparameterizations of the 2-torus, and

$$SL(2) \times GL(27) \times IGL(42) \tag{5.7}$$

is indeed a subgroup of the symmetry of the linear $D = 4$ theory, (5.4). $Sp(56)$ has a subgroup $GL(27) \times SL(2)$ and the $SL(2)$ in (5.7) is a diagonal subgroup of

the $SL(2) \times SL(2)$ in $SL(2) \times Sp(56)$:

$$GL(27) \times SL(2) \subset GL(27) \times SL(2) \times SL(2) \subset Sp(56) \times SL(2) \tag{5.8}$$

The 56 vector gauge fields in $D = 4$ subject to the constraint (5.2) transform as a $(\mathbf{2},\mathbf{27})+(\mathbf{2},\mathbf{1})$ of $SL(2) \times GL(27)$ while the 70 scalar fields transform as a $(\mathbf{1},\mathbf{27})+(\mathbf{42},\mathbf{1})+(\mathbf{1},\mathbf{1})$ of $GL(42) \times GL(27)$. The graviton and its duals transform under $SL(2)$ as in (4.18), subject to the constraints (3.67).

Thus for the free (4,0) theory, reducing on a 2-torus gives a $D = 4$ theory with $SL(2)$ symmetry, and this is indeed a subgroup of the symmetries of the linearised $N = 8$ supergravity theory. In particular, the 56 vector potentials transforming as a **56** of $Sp(56)$ decompose into a $(\mathbf{2}, \mathbf{27}) + (\mathbf{2}, \mathbf{1})$ under the $SL(2) \times GL(27)$ subgroup of $Sp(56)$. In the full non-linear supergravity theories, the interactions break the $GL(27)$ in $D = 5$ to E_6, with the 27 $D = 5$ vector fields transforming as a **27**, and break the $Sp(56)$ in $D = 4$ to E_7, with the 28+28 $D = 4$ vector fields transforming as a **56**. However, although $SL(2) \times GL(27)$ is a subgroup of $Sp(56)$, $SL(2) \times E_6$ is not a subgroup of E_7. As E_7 is the maximal symmetry of the vector/scalar sector of the $D = 4, N = 8$ supergravity theory, this $SL(2)$ cannot be a symmetry of the supergravity equations of motion, and at most an abelian subgroup $\mathbb{R}^+$ can survive, as $\mathbb{R}^+ \times E_6$ is a maximal subgroup of E_7. Moreover, the $SL(2)$ arising from the torus reparameterisations necessarily acts on the graviton, while the usual U-duality symmetries leave the Einstein frame metric invariant, and there is no such symmetry of the non-linear Einstein equations. This $SL(2)$ symmetry is then not a symmetry of the $D = 4, N = 8$ supergravity field equations. This raises a number of issues for the interacting theory, which will be addressed in the next section.

Note that E_7 does have a maximal subgroup $SL(2) \times SO(6, 6)$ which is the product of an $SL(2)$ S-duality and an $SO(6, 6)$ T-duality under which the 56 vector potentials consist of the 24 in the NS-NS sector transforming as a $(\mathbf{2},\mathbf{12})$, while there are 32 in the RR sector transforming as a $(\mathbf{1},\mathbf{32})$, so that these are singlets

under $SL(2)$ [2]. For the $SL(2)$ arising from the reduction of the (4,0) theory on T^2, all the vector fields would be in $SL(2)$ doublets.

6. Symmetries of Interacting Theories

Consider first the (2,0) theory, compactified on a 2-torus to give a $D = 4$ theory with 16 supersymmetries and an $SL(2,\mathbb{Z})$ symmetry arising from the large diffeomorphisms of the torus. In the free case, the (2,0) theory is well-understood and the compactification is straightforward, and the resulting $D = 4$ theory is the $N = 4$ abelian gauge theory which indeed has an $SL(2,\mathbb{Z})$ S-duality symmetry of the equations of motion. In the interacting case, the $D = 6$ (2,0) theory is only known implicitly and a natural candidate for the compactified theory is $N = 4$ non-abelian Yang-Mills theory. However, the $N = 4$ Yang-Mills field theory does not have an $SL(2,\mathbb{Z})$ symmetry of the equations of motion, due to the presence of explicit vector potentials in the minimal couplings. There is some evidence that the full theory should have an $SL(2,\mathbb{Z})$ symmetry – for example, in the holographic description of the $N = 4$ conformal field theory, this arises from the $SL(2,\mathbb{Z})$ U-duality symmetry of the type IIB superstring. The field theory cannot then give a complete formulation of the theory. Indeed, as one approaches the conformal point in moduli space, the W-bosons become massless and so do the magnetic monopoles and the infinite tower of dyons related to the W-boson by S-duality [9]. These light states are mutually non-local and there is no conventional local field theory that can describe all the light degrees of freedom. At the conformal point, there are no particle states and one instead discusses correlation functions of physical operators; see e.g. [13]. Thus in the interacting case, the fact that there is no $SL(2,\mathbb{Z})$ symmetry of the non-abelian super-Yang-Mills field equations can be taken not as an argument against the symmetry, but as an argument against the formulation as a conventional field theory.

The (4,0) theory is superficially similar. The free theory in a flat background can be compactified on a 2-torus to give linearised $N = 8$ supergravity in $D = 4$

which indeed has an $SL(2,\mathbb{Z})$ symmetry inherited from the 2-torus diffeomorphisms. The full non-linear $D = 4$ supergravity theory has local interactions which break the $SL(2,\mathbb{Z})$ duality symmetry; in the gauge theory case, S-duality would transform the gauge potential A into its dual $\tilde{A}$, and the minimal couplings are clearly not invariant, while in the linearised gravity theory the graviton h_{mn} is transformed into the dual gravitons $\hat{h}_{mn}, \tilde{h}_{mn}$ and this symmetry is violated by the non-linear couplings to the metric in the full theory.

Either the full $D = 4$ theory has such an $SL(2,\mathbb{Z})$ symmetry, or it does not. If the full theory does indeed have the $SL(2,\mathbb{Z})$ symmetry, then the situation would be similar to the case of the interacting (2,0) theory, and the fact that the $N = 8$ supergravity theory doesn't have the requisite symmetry would be taken not as evidence against the symmetry, but as an indication that the supergravity doesn't give a complete description of the theory, just as the super-Yang-Mills theory doesn't give a complete description of the theory obtained by reducing the (2,0) theory.

For gravity in any dimension, the gauge symmetry is

$$\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} \tag{6.1}$$

If the metric is written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{6.2}$$

in terms of a fluctuation $h_{\mu\nu}$ about some background metric $\bar{g}_{\mu\nu}$ (e.g. a flat background metric) then two main types of symmetry emerge. The first consists of 'background reparameterizations'

$$\delta\bar{g}_{\mu\nu} = 2\bar{\nabla}_{(\mu}\xi_{\nu)}, \qquad \delta h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} \tag{6.3}$$

where $\bar{\nabla}$ is the background covariant derivative with connection constructed from $\bar{g}_{\mu\nu}$, while $h_{\mu\nu}$ transforms as a tensor ($\mathcal{L}_\xi$ is the Lie derivative with respect to the

vector field ξ), as do all other covariant fields. The second is the 'gauge symmetry' of the form

$$\delta\bar{g}_{\mu\nu} = 0, \qquad \delta h_{\mu\nu} = 2\nabla_{(\mu}\zeta_{\nu)} \tag{6.4}$$

in which $h_{\mu\nu}$ transforms as a gauge field and the background is invariant. There is in addition the standard shift symmetry under which

$$\delta\bar{g}_{\mu\nu} = \alpha_{\mu\nu}, \qquad \delta h_{\mu\nu} = -\alpha_{\mu\nu} \tag{6.5}$$

Combining (6.3) with the shift symmetry (6.5) gives (6.4) (or forms of the transformation that are intermediate between (6.3) and (6.4)), and in terms of the full metric $g_{\mu\nu}$, there is no shift symmetry and a unique gauge symmetry (6.1); the various types of symmetry (6.3),(6.4),(6.5) are an artifice of the background split. The shift symmetry is a signal of background independence and plays an important role in the interacting theory.

The linearised $D = 5$ supergravity theory has both background reparameterization and gauge invariances given by the linearised forms of (6.3),(6.4) respectively, and both of these have origins in $D = 6$ symmetries of the free (4,0) theory. The background reparameterization invariance lifts to the linearised $D = 6$ background reparameterization invariance

$$\delta\bar{g}_{MN} = 2\partial_{(M}\xi_{N)}, \qquad \delta C_{MNPQ} = \mathcal{L}_\xi C_{MNPQ} \tag{6.6}$$

with the transformations leaving the flat background metric $\bar{g}_{MN}$ invariant forming the $D = 6$ Poincaré group. The $D = 5$ gauge symmetry given by the linearised form of (6.4) arises from the $D = 6$ gauge symmetry

$$\delta C_{MN\,PQ} = \partial_{[M}\chi_{N]PQ} + \partial_{[P}\chi_{Q]MN} - 2\partial_{[M}\chi_{NPQ]} \tag{6.7}$$

with $\delta\bar{g}_{MN} = 0$ and the parameters related by $\zeta^\mu = \chi^{55\mu}$. The $D = 6$ theory has no analogue of the shift symmetry, and the emergence of that symmetry on

reduction to $D = 5$ (and dualising to formulate the theory in terms of a graviton $h_{\mu\nu}$) comes as a surprise from this viewpoint. On reducing on a 2-torus, the group $SL(2, \mathbb{Z})$ of large background diffeomorphisms of the 2-torus background give rise to a symmetry of the resulting $D = 4$ theory, just as it would for the reduction of any conventional field theory on a 2-torus.

The gravitational interactions of the full supergravity theory in $D = 5$ are best expressed geometrically in terms of the total metric $g_{\mu\nu}$. If an interacting form of the (4,0) theory exists that reduces to the $D = 5$ supergravity, it must be of an unusual type. One possibility is that there is no background metric of any kind in $D = 6$, and the full theory is formulated in terms of a total field corresponding to C, with a spacetime metric emerging only in a particular background C field and a particular limit corresponding to the free theory limit in $D = 5$.

It is not even clear that such an interacting theory should be formulated in a $D = 6$ spacetime. In $D = 5$, the diffeomorphisms can be taken to act actively, leaving spacetime coordinates invariant and transforming the fields, or passively with the coordinates transforming as

$$\delta x^{\mu} = \xi^{\mu} \tag{6.8}$$

In the (4,0) theory, the parameter ξ^{μ} lifts to a parameter χ^{MNP}. The active diffeomorphisms could lift to transformations of fields in a $D = 6$ spacetime manifold,as in (6.7), but if the passive ones were to lift, it could be to something like a manifold with coordinates X^{MNP} transforming through reparameterisations

$$\delta X^{MNP} = \chi^{MNP} \tag{6.9}$$

with the $D = 5$ spacetime arising as a submanifold with $x^{\mu} = X^{55\mu}$. If this was the case, the picture of a $D = 6$ theory compactified on a 2-torus would only emerge in the free limit, and the lack of an $SL(2, \mathbb{Z})$ symmetry in the interacting $D = 4$ theory would reflect the absence of a conventional spacetime picture in the theory that emerges in the strong coupling limit of the $D = 5$ theory.

Similar considerations apply to the local supersymmetry transformations. In $D=5$, the local supersymmetry transformations in a supergravity background give rise to 'background supersymmetry transformations' with symplectic Majorana spinor parameters $\epsilon^{\alpha a}$ (where α is a $D=5$ spinor index and $a=1,..,8$ labels the 8 supersymmetries) in which the gravitino fluctuation ψ_μ^a transforms without a derivative of ϵ^a, and 'gauge supersymmetries' with spinor parameter $\varepsilon^{\alpha a}$ under which

$$\delta\psi_\mu^a = \partial_\mu \varepsilon^a + \dots \tag{6.10}$$

The background symmetries preserving a flat space background form the $D=5$ super-Poincaré group. In the free theory, the $D=5$ super-Poincaré symmetry lifts to part of a $D=6$ super-Poincaré symmetry with $D=5$ translation parameters ξ^μ lifting to $D=6$ ones Ξ^M and supersymmetry parameters ϵ lifting to $D=6$ spinor parameters $\hat\epsilon$. The corresponding $D=6$ supersymmetry charges Q and momenta P form part of the (4,0) super-Poincaré algebra with

$$\{Q_\alpha^a, Q_\beta^b\} = \Omega^{ab}\big(\Pi_+\Gamma^M C\big)_{\alpha\beta} P_M \tag{6.11}$$

where $\Pi_\pm$ are the chiral projectors

$$\Pi_\pm = \frac{1}{2}(1 \pm \Gamma^7) \tag{6.12}$$

α, β are $D=6$ spinor indices and $a,b=1,\dots,8$ are $USp(8)$ indices, with Ω^{ab} the $USp(8)$-invariant anti-symmetric tensor. This is in turn part of the $D=6$ superconformal group $OSp^*(8/8)$.

The $D=5$ gravitini lift to fermionic 2-form gauge fields $\psi_{MN}^{\alpha a}$ which are also $D=6$ Weyl spinors ($\psi_{MN}^\alpha = \psi_{[MN]}^\alpha$). The supersymmetry (6.10) lifts to the gauge symmetry

$$\delta\psi_{MN}^a = \partial_{[M}\varepsilon_{N]}^a + \dots \tag{6.13}$$

with parameter a spinor-vector $\varepsilon_N^{\alpha a}$. The field strength

$$\chi^{\alpha}_{MNP} = \partial_{[P}\psi^{\alpha}_{MN]} \tag{6.14}$$

satisfies the self-duality constraint

$$\chi^{\alpha a}_{MNP} = \frac{1}{6}\epsilon_{MNPTUV}\chi^{TUV\alpha a} \tag{6.15}$$

The $D = 5$ gauge symmetries including those with parameters $\zeta^{\mu}, \varepsilon^{a}$ satisfy a local algebra whose global limit is the $D = 5$ Poincaré algebra, but the $D = 6$ origin of this (at least in the free theory) is an algebra including the generators $\mathcal{Q}^{a}_{\alpha M}$ of the fermionic symmetries with parameter ε^{α}_{N} and the generators $\mathcal{P}^{MNP}$ of the bosonic symmetries with parameter χ^{MNP}. The global algebra is of the form

$$\{\mathcal{Q}^{a}_{\alpha N}, \mathcal{Q}^{b}_{\beta P}\} = \Omega^{ab}\big(\Pi_{+}\Gamma^{M}C\big)_{\alpha\beta}\mathcal{P}_{(NP)M} \tag{6.16}$$

In the reduction to $D = 5$, the $D = 5$ superalgebra has charges $Q^{a}_{\alpha} = \mathcal{Q}^{a}_{\alpha 5}$, $P_{\mu} = \mathcal{P}_{55\mu}$.

Supersymmetry provides a further argument against the possibility of a background metric playing any role in an interacting (4,0) theory in $D = 6$. The $D = 5$ supergravity can be formulated in an arbitrary supergravity background, but these cannot be lifted to $D = 6$ (4,0) backgrounds involving a background metric as there is no (4,0) multiplet including a metric or graviton. The absence of a (4,0) supergravity multiplet appears to rule out the possibility of a background metric and the standard supersymmetry (6.11) playing any role in the $D = 6$ theory. Indeed, the interacting theory (if it exists) should presumably be a theory based on something like the algebra (6.16) rather than the super-Poincaré algebra (6.11).

There seem to be three main possibilities. The first is that there is no interacting version of the (4,0) theory, that it only exists as a free theory, and that the limit proposed in [1] only exists for the free $D = 5$ theory. The second is that an

interacting form of the theory does exist in 6 spacetime dimensions, with $D = 6$ diffeomorphism symmetry. The absence of a spacetime metric means that such a generally covariant theory must be of an unusual kind. If such a theory exists, then for a spacetime with the topology $T^2 \times M_4$ for some 4-manifold M_4, then the group $SL(2, \mathbb{Z})$ of large diffeomorphisms of the torus should give rise to an S-duality of the dimensionally reduced theory. The $N = 8$ supergravity has no such symmetry, so the reduction would give not the supergravity but some modification of this theory (presumably not both covariant and local) which does have the invariance. This would be similar to the reduction of the interacting (2,0) theory, which gives a $D = 4$ theory with $SL(2, \mathbb{Z})$ invariance, and which therefore cannot be the $N = 4$ super-Yang-Mills field theory. Such a symmetry would imply that $D = 4$ gravity is self-dual, with its strong coupling behaviour being governed by an identical theory to the weak coupling theory.

The third and perhaps the most interesting possibility is that the theory that reduces to the interacting supergravities in $D = 4, 5$ is not a diffeomorphism-invariant theory in six spacetime dimensions, but is something more exotic, perhaps of the type suggested above. If so, there is no obvious reason to expect an $SL(2, \mathbb{Z})$ symmetry in the full theory, but it could arise as a 'bonus symmetry' in the free limit in which the new theory reduces to the free (4,0) field theory.

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